$$S=i\sin3\theta+3i\sin5\theta+6i\sin7\theta+\ldots+\frac{(n-2)(n-4)}{8}i\sin(n-3)\theta,$$

where $\theta=\pi/n$, $i=\sqrt{-1}$.

$$\therefore C+S=\cos3\theta+i\sin3\theta+3(\cos5\theta+i\sin5\theta)+6(\cos7\theta+i\sin7\theta)+\ldots$$

$$+\frac{(n-2)(n-4)}{8}[\cos(n-3)\theta+i\sin(n-3)\theta]=(\cos\theta+i\sin\theta)^3+3(\cos\theta+i\sin\theta)^5$$

$$+6(\cos\theta+i\sin\theta)^7+\ldots+\frac{(n-2)(n-4)}{8}(\cos\theta+i\sin\theta)^{n-3}=e^{3i\theta}+3e^{5i\theta}+6e^{7i\theta}+\ldots$$

$$+\frac{(n-2)(n-4)}{8}e^{(n-3)i\theta}=y^3[1+3y^2+6y^4+10y^6+\ldots+\frac{(n-2)(n-4)}{8}y^{n-6}]$$

$$=\frac{y^3}{(1-y^2)^3}-\frac{(n-\ )(n-4)y^{n+3}+n(n-2)y^{n-1}-2n(n-4)y^{n+1}}{8(1-y^2)^3}$$

Putting $y=e^{i\theta}=\cos\theta+i\sin\theta$, and equating C=the rational part, we get

$$C=\frac{2n(n-4)\sin(2\pi/n)-n(n-2)\sin(4\pi/n)}{64[\sin(\pi/n)]^3}.$$

MISCELLANEOUS.

164. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Find the number of real roots of the equation $100\sin x=x$, and show the largest root is approximately 96.10. Find tan39° to three places of decimals. How many real roots of $\tan x=1/x^2$ lie between 0 and 2π?

Solution by HENRY HEATON, Belfield, N. D.

(1). The equation may be written $x/\sin x=100$, As x varies from 0 to $\frac{1}{2}\pi$, $x/\sin x$ takes the successive values between 1 and $\frac{1}{2}\pi$. Hence the equation has no real root in the first quadrant. As x varies from $\frac{1}{2}\pi$ to π, $x/\sin x$ takes all the successive values from $\frac{1}{2}\pi$ to ∞. Hence the equation has one real root in the second quadrant. In the third and fourth quadrants $x/\sin x$ is negative. Hence there can be no real root. At every round as x passes through the first and second quadrant $x/\sin x$ varies from ∞ to a minimum value then back to ∞. This minimum value is in the first quadrant when $x=\tan x$. After the first round as long as $x<100$ there are two real positive roots to every round. Hence there can be no real positive root when $x>31\pi$, and the whole number of real positive roots is 31. The number of real negotive roots is evidently the same with the same numerical values. Hence the whole number of real roots is 62. The largest real root evidently occurs between $x=30\frac{1}{2}\pi$ and 31π. Put $x=y+30\frac{1}{2}\pi$. Then $\sin(y+30\frac{1}{2}\pi)=\frac{1}{100}(y+30\frac{1}{2}\pi)$.

$\therefore \cos y=.958185+\frac{1}{100}y$.

If $\cos y=.95818$, $y=16°\ 38'=.2903^{(r)}$, a first approximate value of y. If $\cos y=.95818+.002903=.96108$, $y=16°\ 2'=.27983^{(r)}$, a second approximation.

If $\cos y = .95818 + .002798 = .96098$, $y = 16° \ 3\frac{1}{2}' = .28027^{(r)}$, a third approximation. If $\cos y = .95818 + .00281 = .96099$, $y = 16° \ 3\frac{3}{8}' = .28023^{(r)}$, a fourth approximation. Whence $x = 96.098$, nearly.

(2). From the well known length of the side of the inscribed decagon we have $\sin 18° = \cos 72° = \frac{1}{4}(\sqrt{5} - 1)$.

$$\text{Hence } \cos 36° = \sqrt{\frac{1 + \frac{1}{4}(\sqrt{5}-1)}{2}} = \tfrac{1}{4}\sqrt{(6+2\sqrt{5})},$$

$$\text{and } \sin 36° = \sqrt{\frac{1 - \frac{1}{4}(\sqrt{5}-1)}{2}} = \tfrac{1}{4}\sqrt{(10-2\sqrt{5})}.$$

$$\text{Hence } \cot 36° = \tan 54° = \sqrt{\frac{6+2\sqrt{5}}{10-2\sqrt{5}}} = \sqrt{(1+\tfrac{2}{5}\sqrt{5})},$$

$$\tan 15° = \frac{1-\cos 30°}{\sin 30°} = 2 - \sqrt{3}.$$

$$\therefore \tan 39° = \tan(54° - 15°) = \frac{\sqrt{(1+\frac{2}{5}\sqrt{5})} - 2 + \sqrt{3}}{1 + (2-\sqrt{3})\sqrt{(1+\frac{2}{5}\sqrt{5})}}$$

$$= \frac{\sqrt{(5+2\sqrt{5})} - 2\sqrt{5} + \sqrt{15}}{\sqrt{5} + 2\sqrt{(5+\sqrt{5})} - \sqrt{(15+6\sqrt{5})}}.$$

This can be readily computed to any desired degree of accuracy.

(3). As x varies from 0 to $\frac{1}{2}\pi$, $x^2 \tan x$ passes through all successive values from 0 to ∞. Hence, it has one value at which $x^2 \tan x = 1$. The same is true as x varies from π to $3\pi/2$. But in the second and fourth quadrants all values of $x^2 \tan x$ are negative. Hence the equation has no real roots in those quadrants. Hence the number of real roots between 0 and 2π is 2.

Also solved by G. B. M. Zerr and A. H. Holmes. Dr. Zerr finds $\tan 39° = .809785$ by means of $\tan 30°$ and $\tan 9°$. In (3), he finds $x = 51° \ 17\frac{1}{2}'$ and $185° \ 27' \ 10''$.

PROBLEMS FOR SOLUTION.

ALGEBRA.

279. Proposed by THEODORE L. DE LAND, Treasury Department, Washington, D. C.

The United States Panama Canal Bonds were issued, to date August 1, 1906, and will mature on August 1, 1936; and they bear interest at the rate of 2% per annum, payable quarterly, on the first day of November, 1906, and the first day of February, May, and August, 1907, and so on for each succeeding quarter, until the bonds mature, when the principal will be paid at par with the last quarter's interest. The coupon bonds of this loan were quoted on the New York Stock Exchange, at 10.30 a. m., on December 17, 1906, at $103\frac{3}{4}$ bid and $104\frac{3}{4}$ asked.

Required: The rate of interest per annum, payable quarterly, an investor would *realize* if he purchased the Panama bonds on December 17, 1906, and could reinvest his interest income, quarterly, at the *realized* rate.